

THE KELLER-OSSERMAN PROBLEM FOR THE K-HESSIAN OPERATOR

DRAGOS-PATRU COVEI¹

ABSTRACT. A delicate problem is to obtain existence of positive solutions to the boundary blow-up elliptic equation

$$\sigma_k^{1/k}(\lambda(D^2u)) = g(u) \text{ in } \Omega, u = +\infty \text{ on } \partial\Omega$$

where $\sigma_k^{1/k}(\lambda(D^2u))$ is the k -Hessian operator and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Our goal is to provide a necessary and sufficient condition on g to ensure existence of at least one explosive k -admissible positive solution. The main tools for proving existence are the comparison principle and the method of sub and supersolutions.

1. INTRODUCTION

Let Ω be a bounded smooth domain in \mathbb{R}^N ($N \geq 2$) with $\partial\Omega \in C^{4+\alpha}$ ($\alpha \in (0, 1)$) and such that the curvature of $\partial\Omega$ satisfies $k_m[\partial\Omega] > 0$ for $1 \leq m \leq N-1$ (cf. Ivochkina [24, (Definition 2.4., p. 83)]) and D^2u be the Hessian matrix of a C^2 (i.e., a twice continuously differentiable) function u defined over Ω and $\lambda(D^2u) = (\lambda_1, \dots, \lambda_N)$ be the vector of eigenvalues of D^2u . For $k = 1, 2, \dots, N$ define the k -Hessian operator as follows

$$\sigma_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \cdot \dots \cdot \lambda_{i_k}$$

as the k^{th} elementary symmetric polynomial of the Hessian matrix of u (see for details the important works of [20]-[25]). In other words, $\sigma_k(\lambda(D^2u))$ it is the sum of all $k \times k$ principal minors of the Hessian matrix D^2u and so it is a second order differential operator, which may also be called the k -trace of D^2u denoted also by

$$T_k[u] := tr_k u_{xx}$$

where u_{xx} is the Hesse matrix. We would like to mention that

$$\sigma_1(\lambda(D^2u)) = \sum_{i=1}^N \lambda_i = \Delta u$$

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*Corresponding author.

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is the well known classical Laplace operator and

$$\sigma_N (\lambda (D^2 u)) = \prod_{i=1}^N \lambda_i = \det (D^2 u)$$

is the Monge-Ampère operator. Then, for $k \geq 2$ we know that the k -Hessian operator is a fully nonlinear partial differential operator of divergence form. Denote the set of k -admissible functions in Ω by

$$\Phi^k (\Omega) = \{u \in C^2 (\Omega) \mid \sigma_k (\lambda (D^2 u)) > 0 \text{ for } k = 1, 2, \dots, N\}.$$

The main goal of this paper is to study the existence of solutions of the following fully nonlinear, second order partial differential equation with boundary blow-up of the form

$$\begin{cases} \sigma_k^{1/k} (\lambda (D^2 u)) = \sigma_k^{1/k} (\lambda) = g(u) & \text{in } \Omega \\ \lim_{x \rightarrow x_0} u(x) = +\infty & \forall x_0 \in \partial\Omega \end{cases} \quad (1.1)$$

where $k \in \{1, 2, \dots, N\}$ and $g : [0, \infty) \rightarrow [0, \infty)$ is a function which satisfies:

(G1) g is convex, monotone non-decreasing, $g^k \in C^{2+\alpha}([0, \infty), [0, \infty))$ with $\alpha \in (0, 1)$, $g(0) = 0$ and $g(s) > 0$ for all $s > 0$;

(G2) there exists $\beta > 0$ such that

$$\int_{\beta}^{\infty} \frac{1}{\sqrt[k+1]{(k+1)(G(t) - G(\beta))}} dt < \infty \text{ for } G(t) = \int_0^t g^k(z) dz.$$

The problem (1.1) belongs to the class of fully nonlinear elliptic equations and it is closely related to a geometric problem (see [36], [37]). Hence, the k -Hessian operator appears naturally and it is not introduced as a straightforward generalization of the Laplace or Monge-Ampère operator.

The study of existence of large solutions for semilinear elliptic systems of the form (1.1) goes back to the pioneering papers by Osserman [27] and Keller [31, 32]. In fact, from the results of [31] and [27] we know that, for a given positive, continuous and nondecreasing function g and a bounded domain $\Omega \subset \mathbb{R}^N$, the semilinear elliptic partial differential equation $\Delta u = g(u)$ in Ω , possesses a large solution $u : \Omega \rightarrow \mathbb{R}$ if and only if the nowadays called Keller-Osserman condition holds, i. e.

$$\int_1^{\infty} \left(\int_0^t g(s) ds \right)^{-1/2} dt < +\infty. \quad (1.2)$$

In the present work we will limit ourselves to the development of mathematical theory for the more general problem (1.1). In our direction, but for the special case $k = 1$ or $k = N$, there are many papers dealing with existence, uniqueness and asymptotic behavior issues for blow-up solutions of (1.1). Here we wish to mention the works of Diaz [9], Osserman [27], Matero [28, 29], Pohozaev [30] and Keller [31] (see also references therein). However, to the best of our knowledge, excepting the case $\Omega = \mathbb{R}^N$ studied by Ji-Bao [1] and Bao-Ji-Li [2], we don't know any results about the existence of solutions for the general problem (1.1), that so naturally appears in geometry referred as k -Yamabe problem.

Therefore, in contrast to numerous results on the case $k = 1$ less is known about the situation $k \in \{2, \dots, N\}$, since the situation $k \in \overline{2, N}$ is not so straightforward.

But, a possible starting point to approach this kind of problems could be works such as [1], [6], [31], [27], [29] and [33].

We begin by stating our result on existence of solutions.

Theorem 1.1. *Let $g : [0, \infty) \rightarrow [0, \infty)$ be a function satisfying (G1). Then, g satisfy the Keller-Osserman type condition (G2) if and only if the problem (1.1) admits at least one explosive solution u in any bounded smooth domain in \mathbb{R}^N ($N \geq 2$) with $\partial\Omega \in C^{4+\alpha}$ and the curvature of $\partial\Omega$ satisfies $k_m[\partial\Omega] > 0$ for $1 \leq m \leq N - 1$.*

Since there is a proof of Theorem 1.1 for equation (1.1) in [6], [28] for the case $k = 1$, it will be taken to be known in what follows.

The contributions of our paper are:

- 1.: We established a necessary and sufficient condition for the nonlinearity g such that the considered boundary blowup k -Hessian equation has solution. The sufficient part has been already obtained by other authors but for particular nonlinearities g that are in C^∞ .
- 2.: Our methodology is new and it can be applied for more general nonlinearities depending on the regularity results obtained for the k -Hessian operator. The necessary part is proved by analyzing the radially symmetric solution of the considered equation.

The reminder of this paper is organized as follow. The Section 2 is devoted to the presentation of some basic results which are needed for the study of the positive solutions for (1.1). Part of the results will be fully proven, and, for some of them, only the statements will be exposed. Section 1.1 contains the proof of the main result.

2. PRELIMINARIES

Let $\{k_i\}_1^{n-1}$ be the set of principal curvatures of $\partial\Omega$ at x and $\sigma_{k-1}(k_1, \dots, k_{n-1})$ be the $(k-1)$ -curvature of $\partial\Omega$ and $B_R \subset \Omega$ be a ball of radius $R > 0$. We extract from [25, pp. 12] (see also [3, Theorem 3, p. 264]) the following result.

Lemma 2.1. *The Dirichlet problem*

$$\begin{cases} \sigma_k^{1/k}(\lambda(D^2u)) = \psi > 0 \text{ in } \overline{\Omega}, k > 1, \\ u = \text{const on } \partial\Omega \end{cases} \quad (2.1)$$

admits a (unique) admissible solution $u \in \Phi^k(\Omega)$ provided that $\partial\Omega$ is connected, and

$$\text{at every point } x \in \partial\Omega, \sigma_{k-1}(k_1, \dots, k_{n-1}) > 0. \quad (2.2)$$

We need to mention that in Lemma 2.1 the fact that $\sigma_{k-1}(k_1, \dots, k_{n-1}) > 0$ it means that $\partial\Omega$ is indicated as $(k-1)$ -convex and by convention we have $\sigma_0(\lambda) := 1$. Now, an argument similar to [4, (Definition 2.2, Remark 2.3)] leads to the following definition.

Definition 2.2. Let g be a continuous function on \mathbb{R} and $c \in \mathbb{R}_+^*$. A subsolution of

$$\begin{cases} \sigma_k^{1/k}(\lambda(D^2u)) = g(u) \text{ in } \Omega, \\ u = c \text{ on } \partial\Omega, \end{cases} \quad (2.3)$$

is a function $\underline{u} : \Omega \rightarrow \mathbb{R}$ from $\Phi^k(\Omega)$ such that

$$\begin{cases} \sigma_k^{1/k}(\lambda(D^2\underline{u}(x))) \geq g(\underline{u}(x)) & \text{for all } x \in \Omega, \\ \underline{u} \leq c & \text{on } \partial\Omega. \end{cases} \quad (2.4)$$

Similarly, a supersolution of (2.3) is a function $\overline{u} : \Omega \rightarrow \mathbb{R}$ from $C^2(\Omega)$ such that

$$\begin{cases} \sigma_k^{1/k}(\lambda(D^2\overline{u}(x))) \leq g(\overline{u}(x)) & \text{for all } x \in \Omega, \\ \overline{u} \geq c & \text{on } \partial\Omega. \end{cases} \quad (2.5)$$

Finally, u is said to be a solution of (2.3) if u is a subsolution and a supersolution of (2.3).

The following variant of the comparison principle will be used. The proof of the result goes as in [9, (Proposition 2.43, p. 187)] (or: Jian [26], [15, (Lemma 2.3, p. 249)]).

Lemma 2.3. *Assume that $g : [0, \infty) \rightarrow [0, \infty)$ satisfies condition (G1), $w_1 \in \Phi^k(\Omega)$ is a subsolution of (2.3) and that $w_2 \in C^2(\Omega)$ is a supersolution of (2.3). If $w_1 \leq w_2$ on $\partial\Omega$ then $w_1 \leq w_2$ in Ω .*

We state and prove a version of the method of sub and supersolutions. The existence part of the following Lemma is inspired by the paper of [14], [34] and [35].

Lemma 2.4. *Let c be a positive constant. Assume that $g : [0, \infty) \rightarrow [0, \infty)$ satisfies (G1) and that there exists a subsolution $\underline{u} \in \Phi^k(\Omega)$ (resp. a supersolution $\overline{u} \in C^2(\Omega)$) of (2.3). Under these hypotheses, the problem*

$$\sigma_k^{1/k}(\lambda(D^2u)) = g(u) \text{ in } \Omega, \quad u|_{\partial\Omega} = c \quad (2.6)$$

possesses a unique k -admissible solution $u \in C^{2+\alpha}(\Omega)$ (with $\alpha \in (0, 1)$) such that $\underline{u} \leq u \leq \overline{u}$ in Ω .

Proof. We are showing that for every positive constant c , the problem (2.6) admits a unique positive solution $u \in \Phi^k(\Omega)$. Let $\Lambda < 0$ such that

$$-\Lambda \geq \frac{g(v) - g(w)}{v - w}$$

for every v, w with $\underline{u} \leq w < v \leq \overline{u}$. Starting with $u_0 = \underline{u}$ we inductively define a sequence $\{u_j\}_{j \geq 1}$ such that

$$\begin{cases} \sigma_k(\lambda(D^2u_j)) + \Lambda u_j = g^k(u_{j-1}) + \Lambda u_{j-1} & \text{in } \Omega \\ u_j = c & \text{on } \partial\Omega. \end{cases}$$

By comparison principle, and Lemma 2.3, we have

$$\underline{u} \leq u_1 \leq u_2 \leq \dots \leq u_{j-1} \leq u_j \leq \dots \leq \overline{u} \text{ on } \overline{\Omega}.$$

Thus, the sequence $\{u_j\}_{j \geq 1}$ is increasing and bounded by some constants independent of j which implies that there exists

$$u(x) = \lim_{j \rightarrow \infty} u_j(x) \text{ for } x \in \overline{\Omega}.$$

Then

$$\int_{\Omega'} (g^k(u_{j-1}) + \Lambda u_{j-1}) \varphi(x) dx \xrightarrow{j \rightarrow \infty} \int_{\Omega'} (g^k(u) + \Lambda u) \varphi(x) dx$$

for any Borel measurable $\Omega' \subseteq \Omega$ and any $\varphi \in C_0^\infty(\Omega)$, where we have used the divergence form of the k -Hessian and the condition g is convex imposed in [34, 35]. By standard estimates (see for example [3] or [14]) we find that $u \in C^{2+\alpha}(\Omega)$ with $\alpha \in (0, 1)$ is a solution of (2.6) which, thanks to the monotonicity of g and the comparison principle, Lemma 2.3, it is unique.

The existence of a subsolution \underline{u} and a supersolution \bar{u} from Lemma 2.4 is given in the following Lemma.

Lemma 2.5. *If $g : [0, \infty) \rightarrow [0, \infty)$ satisfies (G1) then:*

i) *the solution $\bar{u} \in C^2(\Omega)$ of problem*

$$\begin{cases} \Delta \bar{u} = N \left[(C_N^k)^{-1} g(\bar{u}(x)) \right]^{1/k}, & x \in \Omega, \\ \bar{u}(x) \rightarrow \infty & \text{as } x \rightarrow \partial\Omega, \end{cases}$$

given by the result of [31], is a supersolution of the problem (2.6) for any positive constant c ;

ii) *the solution $\underline{u} \in \Phi^k(\Omega)$ of the problem*

$$\begin{cases} \sigma_k^{1/k}(\lambda(D^2 \underline{u})) = g(c) > 0 & \text{in } \bar{\Omega}, \quad k > 1, \\ \underline{u} = c & \text{on } \partial\Omega, \end{cases}$$

given by Lemma 2.1, is a subsolution of the problem (2.6) for any positive constant c .

iii) *the subsolution $\underline{u} \in \Phi^k(\Omega)$ and the supersolution $\bar{u} \in C^2(\Omega)$ determined in i) and ii) are such that $\underline{u} \leq \bar{u}$ in $\bar{\Omega}$.*

Proof. i) The Maclaurin's inequalities

$$\frac{1}{N} \Delta \bar{u} \geq \left[(C_N^k)^{-1} \sigma_k(\lambda(D^2 \bar{u})) \right]^{1/k} \quad \text{for any } k = 1, \dots, N, \quad (2.7)$$

where C_N^k is the binomial coefficient, gives

$$N \left[(C_N^k)^{-1} \sigma_k(\lambda(D^2 \bar{u})) \right]^{1/k} \leq \Delta \bar{u} = N \left[(C_N^k)^{-1} g(\bar{u}(x)) \right]^{1/k}.$$

Thus

$$\sigma_k^{1/k}(\lambda(D^2 \bar{u})) \leq g(\bar{u}(x)) \quad \text{in } \Omega.$$

ii) To prove the affirmation we observe that

$$\begin{cases} \sigma_k^{1/k}(\lambda(D^2 \underline{u})) = g(c) \geq g(\underline{u}) & \text{in } \Omega, \\ \underline{u}(x)|_{\partial\Omega} = c|_{\partial\Omega} = c, \end{cases}$$

where we have used the fact that $\underline{u} \leq c$ in Ω . We note that $\underline{u} \leq c$ in Ω is a consequence of Lemma 2.3.

iii) Again, we use the Maclaurin's inequalities

$$\frac{1}{N} \Delta \underline{u} \geq \left[(C_N^k)^{-1} \sigma_k(\lambda(D^2 \underline{u})) \right]^{1/k} \quad \text{for any } k = 1, \dots, N,$$

to see that

$$\frac{1}{N}\Delta \underline{u} \geq \left[(C_N^k)^{-1} g(\underline{u}) \right]^{1/k}. \quad (2.8)$$

If arguing by counterposition $\omega = \{x \in \Omega \mid \underline{u} > \bar{u}\} \neq \emptyset$, then (2.8) becomes

$$\frac{1}{N}\Delta \underline{u} \geq \left[(C_N^k)^{-1} g(\underline{u}) \right]^{1/k} \geq \left[(C_N^k)^{-1} g(\bar{u}) \right]^{1/k} = \frac{1}{N}\Delta \bar{u} \text{ in } \omega$$

from which we have $-\Delta \underline{u} \leq -\Delta \bar{u}$ in ω and therefore, by the classical comparison principle for the laplacian, we obtain that $\underline{u} \leq \bar{u}$ in ω . This is a contradiction with the assumption. The proof is now completed.

Remark 2.6. If the domain Ω is a ball B_R then u is a radial solution (if it is not so, then we can get another solution by rotating u , but we have proved that u is the unique solution). Here, we have used the fact that the k -Hessian operator is invariant with respect to rotations.

For the readers' convenience, we recall the radial form of the k -Hessian operator.

Remark 2.7. (see [1, (Lemma 2.1, p. 178)]) Assume $\varphi \in C^2[0, R]$ is radially symmetric with $\varphi'(0) = 0$. Then, for $k \in \{1, 2, \dots, N\}$ and $u(x) = \varphi(r)$ where $r = |x| < R$, we have that $u \in C^2(B_R)$, and

$$\begin{aligned} \lambda(D^2 u(r)) &= \begin{cases} \left(\varphi''(r), \frac{\varphi'(r)}{r}, \dots, \frac{\varphi'(r)}{r} \right) & \text{for } r \in (0, R), \\ (\varphi''(0), \varphi''(0), \dots, \varphi''(0)) & \text{for } r = 0 \end{cases} \\ \sigma_k(\lambda(D^2 u(r))) &= \begin{cases} C_{N-1}^{k-1} \varphi''(r) \left(\frac{\varphi'(r)}{r} \right)^{k-1} + C_{N-1}^{k-1} \frac{N-k}{k} \left(\frac{\varphi'(r)}{r} \right)^k & \text{for } r \in (0, R), \\ C_N^k (\varphi''(0))^k & \text{for } r = 0, \end{cases} \end{aligned}$$

where the prime denotes differentiation with respect to $r = |x|$.

The proof of the next result can be found in [1, (Lemma 3.1, p. 183)] or [2, (Lemma 12, p. 2154)].

Lemma 2.8. *Let g be a monotone non-decreasing continuous function defined on \mathbb{R} . Let $\xi \in C^2([0, R])$ be a radially symmetric solution of the problem*

$$\begin{aligned} \sigma_k^{1/k}(\lambda(D^2 \xi(r))) &\leq g(\xi(r)) \text{ in } B_R, \\ \xi(r) &\rightarrow \infty \text{ as } r \rightarrow R \end{aligned}$$

where B_R is the ball from \mathbb{R}^N ($N \geq 2$) by radius $R > 0$. Then, if $u \in \Phi^k(\Omega)$ is any bounded solution of

$$\sigma_k^{1/k}(\lambda(D^2 u)) = g(u) \text{ in } \Omega,$$

we have that $u(x) \leq \xi(|x|)$ at each point $x \in B_R$.

The following Lemma is a consequence of results from a number of works, we mention [6].

Lemma 2.9. Assume that $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $g(0) = 0$ and $g(s) > 0$ for $s > 0$. The following statements are equivalent

KO1) there exists $\beta > 0$ such that

$$\mathcal{K}(\beta) := \int_{\beta}^{\infty} \frac{1}{\sqrt[k+1]{(k+1)(G(t) - G(\beta))}} dt < \infty, \quad (2.9)$$

KOL1) the Sharpened Keller-Osserman condition

$$\liminf_{\beta \rightarrow \infty} \mathcal{K}(\beta) = 0. \quad (2.10)$$

The next estimates is almost identical to that of [6].

Lemma 2.10. Assume that $g : [0, \infty) \rightarrow [0, \infty)$ satisfies (G1). If $\xi \in C^2(0, \rho)$ is a non-decreasing function solving

$$C_{N-1}^{k-1} \left[\frac{r^{N-k}}{k} \left(\xi'(r) \right)^k \right]' = r^{N-1} g^k(\xi(r)) \text{ in } (0, \rho), \quad (2.11)$$

then, for $0 < \rho_1 < \rho_2 < \rho$, we have that

$$\int_{\xi(\rho_1)}^{\xi(\rho_2)} \frac{(C_{N-1}^{k-1})^{1/(k+1)}}{\sqrt[k+1]{(k+1)G(\xi(r)) - G(\xi(\rho_1))}} dr \geq \frac{k}{N-2k} \rho_1^{\frac{2k}{k+1}} \left[1 - \left(\frac{\rho_1}{\rho_2} \right)^{\frac{N}{k}-2} \right], \quad (2.12)$$

given that $N \neq 2k$, and

$$\int_{\xi(\rho_1)}^{\xi(\rho_2)} \frac{(C_{N-1}^{k-1})^{1/(k+1)}}{\sqrt[k+1]{(k+1)G(\xi(r)) - G(\xi(\rho_1))}} dr \geq \rho_1^{\frac{2k}{k+1}} \ln \frac{\rho_2}{\rho_1}, \quad (2.13)$$

if $N = 2k$.

Proof. A simple calculation show that for $r \in (\rho_1, \rho_2)$ we have that $\xi'(r) > 0$. Moreover, (2.11) is equivalent to

$$C_{k-1}^{N-1} \xi''(r) \left(\frac{\xi'(r)}{r} \right)^{k-1} + C_{N-1}^{k-1} \frac{N-k}{k} \left(\frac{\xi'(r)}{r} \right)^k = g^k(\xi(r)). \quad (2.14)$$

Multiplying the equation (2.14) by $r^{N+\frac{N}{k}-2} \xi'(r)$, we get that

$$\left[\left(r^{\frac{N}{k}-1} \xi'(r) \right)^{k+1} \right]' = \frac{k+1}{C_{N-1}^{k-1}} g^k(\xi(r)) r^{N+\frac{N}{k}-2} \xi'(r). \quad (2.15)$$

Integrating (2.15) from ρ_1 to r we obtain

$$\begin{aligned} \left(r^{\frac{N}{k}-1} \xi'(r) \right)^{k+1} &\geq \left(r^{\frac{N}{k}-1} \xi'(r) \right)^{k+1} - \left(\rho_1^{\frac{N}{k}-1} \xi'(\rho_1) \right)^{k+1} \\ &= \int_{\rho_1}^r \frac{k+1}{C_{N-1}^{k-1}} g^k(\xi(s)) s^{N+\frac{N}{k}-2} \xi'(s) ds \\ &\geq \frac{k+1}{C_{N-1}^{k-1}} \rho_1^{N+\frac{N}{k}-2} (G(\xi(r)) - G(\xi(\rho_1))). \end{aligned} \quad (2.16)$$

Especially, by (2.16) we have that

$$r^{\frac{N}{k}-1} \xi'(r) \geq \left(\frac{k+1}{C_{N-1}^{k-1}} \right)^{1/(k+1)} \rho_1^{\frac{N}{k}-\frac{2}{k+1}} \sqrt[k+1]{G(\xi(r)) - G(\xi(\rho_1))}.$$

Equivalently, this can be written in the following way

$$\left(\frac{C_{N-1}^{k-1}}{k+1} \right)^{1/(k+1)} \frac{\xi'(r)}{\sqrt[k+1]{G(\xi(r)) - G(\xi(\rho_1))}} \geq \left(\frac{\rho_1}{r} \right)^{\frac{N}{k}-1} \rho_1^{1-\frac{2}{k+1}}. \quad (2.17)$$

Integrating the relation (2.17) from ρ_1 and ρ_2 , we obtain:

given that $2k \neq N$ the relation holds

$$\begin{aligned} & \left(\frac{C_{N-1}^{k-1}}{k+1} \right)^{1/(k+1)} \int_{\xi(\rho_1)}^{\xi(\rho_2)} \frac{1}{\sqrt[k+1]{G(\xi(r)) - G(\xi(\rho_1))}} dt \\ & \geq \int_{\rho_1}^{\rho_2} \left(\frac{\rho_1}{t} \right)^{\frac{N}{k}-1} \rho_1^{1-\frac{2}{k+1}} dt = \rho_1^{\frac{N}{k}-\frac{2}{k+1}} \left(\frac{t^{2-\frac{N}{k}}}{2-\frac{N}{k}} \Big|_{\rho_1}^{\rho_2} \right) \\ & = \frac{k}{N-2k} \rho_1^{\frac{N}{k}-\frac{2}{k+1}} \left(\rho_1^{2-\frac{N}{k}} - \rho_2^{2-\frac{N}{k}} \right), \end{aligned}$$

and the inequality holds

$$\begin{aligned} \left(\frac{C_{N-1}^{k-1}}{k+1} \right)^{1/(k+1)} \int_{\xi(\rho_1)}^{\xi(\rho_2)} \frac{1}{\sqrt[k+1]{G(\xi(r)) - G(\xi(\rho_1))}} dr & \geq \rho_1^{1-\frac{2}{k+1}} \int_{\rho_1}^{\rho_2} \frac{\rho_1}{t} dt \\ & = \rho_1^{2-\frac{2}{k+1}} \ln \frac{\rho_2}{\rho_1}, \end{aligned}$$

if $2k = N$. Thus we get the conclusion.

We can also obtain an estimate as in (2.12)-(2.13) using Maclaurin's inequalities (2.7).

An important consequence of (2.12) and (2.13) is the next:

Lemma 2.11. *Let $g : [0, \infty) \rightarrow [0, \infty)$ be a function satisfying (G1). We have: g satisfy the Keller-Osserman type condition (G2) if and only if the problem (1.1) admits at least one positive $u \in \Phi^k(B_\rho)$ on some ball B_ρ .*

Proof. We deal with the first implication. To prove it we shall proceed as follows. If $2k \neq N$ we assume temporarily that

$$(C_{N-1}^{k-1})^{1/(k+1)} \mathcal{K}(\beta) < \frac{k}{|N-2k|}.$$

Let $\underline{\xi} \in \Phi^k(B_1)$ be the subsolution constructed in Lemma 2.5 with $c = \beta$ and $\bar{\xi} \in C^2(B_1)$ be the supersolution constructed in that way. It is clear that

$$\underline{\xi}(x) \leq \bar{\xi}(x) \text{ for } x \in \bar{\Omega}.$$

An alternative sub and supersolution in the space $\Phi^k(B_1)$ can be obtained as in [1, 2]. From Lemma 2.4 we know that there exists a unique radial solution $\xi \in C^{2,\alpha}(B_1)$ that solves the problem (2.3) with $\Omega = B_1$ and such that $\underline{\xi} \leq \xi \leq \bar{\xi}$. By the knowledge of the classical theory for ordinary differential equations (see, for example [2, (Lemma 9, p. 2148)] or [1]), we know that choosing $\tilde{\beta} = \xi(0)$ and $\xi'(0) = 0$, the solution $\xi(r) := u(x)$ (for $r = |x|$) can be extended to maximal interval $[0, \rho)$ around the point $r_0 = 1$ where $\xi'(r) \neq 0$. Let us point that in the paper of [2] we have all the discussion to understand our problem. Assume that $\rho < \infty$, then u is an explosive solution in the ball B_ρ . Indeed, by the definition of ρ , we have

$$\text{or } \xi(\rho) = +\infty \text{ or } \xi'(\rho) = +\infty.$$

In the case $\xi'(\rho) = +\infty$, integrating between 0 and r in (2.15) we obtain that

$$\left(r^{\frac{N}{k}-1}\xi'(r)\right)^{k+1} = \frac{k+1}{C_{N-1}^{k-1}} \left[G(\xi(r))r^{N+\frac{N}{k}-2} - \left(N + \frac{N}{k} - 2\right) \int_0^r G(\xi(s))s^{N+\frac{N}{k}-3} ds \right]. \quad (2.18)$$

Using the equality (2.18) we get

$$r^{\frac{N}{k}-1}\xi'(r) \leq \left(\frac{k+1}{C_{N-1}^{k-1}}\right)^{1/(k+1)} G^{1/(k+1)}(\xi(r))r^{\frac{N}{k}-\frac{2}{k+1}}, \quad r \in [0, \rho). \quad (2.19)$$

We write Eq. (2.19) in the form

$$\xi'(r) \leq \left(\frac{k+1}{C_{N-1}^{k-1}}\right)^{1/(k+1)} G^{1/(k+1)}(\xi(r))r^{\frac{k-1}{k+1}}, \quad r \in [0, \rho).$$

Then $G(\xi(\rho)) = +\infty$, $\xi(\rho) = +\infty$ and as a conclusion $\xi(r) := u(x)$ is an explosive solution. We next turn to the proof of $\rho < \infty$. In contrary, using Lemma 2.10 with $\rho_1 = 1$ and $\rho_2 > 1$, we observe that

$$(C_{N-1}^{k-1})^{1/(k+1)} \mathcal{K}(\beta) \geq \frac{k}{N-2k} \left[1 - \left(\frac{1}{\rho_2}\right)^{\frac{N}{k}-2} \right],$$

if $N \neq 2k$ and

$$(C_{N-1}^{k-1})^{1/(k+1)} \mathcal{K}(\beta) \geq \ln \rho_2,$$

if $N = 2k$. Using the Lemma 2.9 for ρ_2 approaching ∞ , we obtain a contradiction if either $N = 2k$ or

$$(C_{N-1}^{k-1})^{1/(k+1)} \mathcal{K}(\beta) < \frac{k}{|N-2k|}.$$

In the case $N \neq 2k$ and $\mathcal{K}(\beta) \geq \frac{k}{|N-2k|}$, direct calculation prove that we can choose $C > 0$ sufficiently large such that

$$\frac{1}{C} (C_{N-1}^{k-1})^{1/(k+1)} \mathcal{K}(\beta) < \frac{k}{|N-2k|}.$$

Repeating the above discussion we obtain that if g is replaced by a well determined constant c_1 multiplied with g , then $\tilde{u}(x) := u(x/c_1)$ is an explosive solution of the problem (2.11) with the nonlinear function g in $B_{\rho c_1}$ which denote the concentric

ball of radius ρc_1 . We have already checked that if the Keller-Osserman type condition hold, then there exists some ball in which the solution u blows up to finite value of ρ .

Our next step is to prove the second implication. For this, we assume that there exists some ball B of radius ρ , whose center we may always assume to be the origin, in which $u(x)$ solve the problem (1.1). By the above proof and [2], we may always assume that u is a radial solution and we define $\xi(r) = u(x)$ for $r = |x|$, so that ξ verify the problem (2.11) in $[0, \rho)$. A calculation akin to that in Lemma 2.10 leads to

$$\left[\left(r^{\frac{N}{k}-1} \xi'(r) \right)^{k+1} \right]' = \frac{k+1}{C_{N-1}^{k-1}} g^k(\xi(r)) r^{N+\frac{N}{k}-2} \xi'(r).$$

Integrating this equation from 0 to r , we have

$$\begin{aligned} \left(r^{\frac{N}{k}-1} \xi'(r) \right)^{k+1} &= \int_0^r \frac{k+1}{C_{N-1}^{k-1}} g^k(\xi(z)) z^{N+\frac{N}{k}-2} \xi'(z) dz \\ &\leq \frac{k+1}{C_{N-1}^{k-1}} r^{N+\frac{N}{k}-2} [G(\xi(r)) - G(\xi(0))]. \end{aligned}$$

Evidently,

$$r^{\frac{N}{k}-1} \xi'(r) \leq \left(\frac{k+1}{C_{N-1}^{k-1}} \right)^{1/(k+1)} r^{\frac{N}{k}-\frac{2}{k+1}} {}^{k+1}\sqrt{G(\xi(r)) - G(\xi(0))}.$$

Integrating once more between 0 and ρ , it follows that

$$0 \leq \int_0^\rho \frac{\xi'(r)}{{}^{k+1}\sqrt{(k+1)(G(\xi(r)) - G(\xi(0)))}} dr \leq \frac{(C_{N-1}^{k-1})^{-1/(k+1)} (k+1)}{2k} \rho^{\frac{2k}{k+1}}. \quad (2.20)$$

So the conclusion can be obtained by choosing $\beta = \xi(0)$ in (2.9), which finishes the proof of the lemma.

The following result shows the existence of solutions on small balls.

Lemma 2.12. *Assume that $g : [0, \infty) \rightarrow [0, \infty)$ satisfies (G1). If (2.10) holds and B_ρ is a ball of radius ρ then*

$$\rho_0 = \inf\{\rho > 0 : (1.1) \text{ has a solution in } B_\rho\} = 0. \quad (2.21)$$

It is an easy exercise to prove Lemma 2.12. To do this we use the results in [2] for the k -Hessian operator.

Proof. We assume by contradiction that $R_0 > 0$. Let β_n be a sequence of real numbers increasing to infinity and satisfying

$$\lim_{\beta_n \rightarrow \infty} \mathcal{K}(\beta_n) = 0.$$

Let $\underline{u} = \underline{u}^n \in \Phi^k(B_{\rho_0/2})$ be the radial subsolution obtained in Lemma 2.5 corresponding to $\Omega = B_{\rho_0/2}$ and with $c = \beta_n$ and \bar{u} the supersolution constructed.

It follows in a standard fashion, that for \underline{u} and \bar{u} there exists a unique radial solution in $C^{2,\alpha}(B_{\rho_0/2})$, denoted by u_n , of the problem

$$\begin{cases} \sigma_k^{1/k}(\lambda(D^2 u_n)) = g(u_n) & \text{in } B_{\rho_0/2}, \\ u_n = \beta_n & \text{on } \partial B_{\rho_0/2}. \end{cases}$$

Using the definition of ρ_0 and considering the problem

$$\begin{cases} C_{N-1}^{k-1} \left[\frac{r^{N-k}}{k} (\xi'(r))^k \right]' = r^{N-1} g^k(\xi(r)) & \text{in } (0, \rho), \\ \alpha_n = \xi_n(0) = u_n(0), \xi_n'(0) = 0. \end{cases} \quad (2.22)$$

then the solution $\xi_n(r) := u_n(x)$ for $r = |x|$ of (2.22) can be extended so that remains a solution of (2.22) in $[0, \rho_0]$. Next, apply Lemma 2.10 with $\rho_1 = \rho_0/2$ and $\rho_2 = \rho_0$ to get

$$\int_{\xi(\rho_1)}^{\xi(\rho_2)} \frac{(C_{N-1}^{k-1})^{1/(k+1)}}{k+1 \sqrt{(k+1) G(\xi(r)) - G(\xi(\rho_1))}} dr \geq \frac{k}{N-2k} \rho_0^{\frac{2k}{k+1}} \left[1 - \left(\frac{1}{2} \right)^{\frac{N}{k}-2} \right],$$

when $N \neq 2k$ and

$$\int_{\xi(\rho_1)}^{\xi(\rho_2)} \frac{(C_{N-1}^{k-1})^{1/(k+1)}}{k+1 \sqrt{(k+1) G(\xi(r)) - G(\xi(\rho_1))}} dr \geq \left(\frac{\rho}{2} \right)^{\frac{2k}{k+1}} \ln 2,$$

if $N = 2k$. Passing to the limit as $n \rightarrow \infty$, we obtain a contradiction in both cases.

At this point we can state the following remark.

Remark 2.13. Let u be some positive, k -admissible solution to the equation (1.1) in $\Omega \subset \mathbb{R}^N$. Assume that $\partial\Omega \in C^{4+\alpha}$, the curvature of $\partial\Omega$ satisfies $k_m[\partial\Omega] > 0$ for $1 \leq m \leq N-1$ and conditions (G1) and (G2) are satisfied. Then the inequality

$$\sup_{\Omega'} u(x) \leq c \left(\frac{1}{\rho} \right), \quad \rho = \text{dist} \{ \partial\Omega', \partial\Omega \}$$

holds true.

After these preliminaries we can begin to analyze the problem (1.1).

3. THE PROOF OF THEOREM 1.1

We are devoting this section to prove Theorem 1.1. A natural way to construct solutions in the case $k = 1$ is by solving the finite datum Dirichlet problem and then letting the datum grow to infinity to obtain the conclusion (see also [26]). In the same way, as for the case $k = 1$, we prove the first implication. Let u_n be

the unique solution of the problem

$$\begin{cases} \sigma_k^{1/k}(\lambda(D^2 u)) = g(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = n, \quad n \in \mathbb{N}^*, \end{cases} \quad (3.1)$$

which clearly exists. Then $\underline{u} \leq u_n \leq \bar{u}$ and u_n is non-decreasing. The Lemma 2.11 shows that for any $x \in \Omega$ there exists some ball $B(x, \tau) \subset \Omega$ such that

$$\begin{cases} \sigma_k^{1/k}(\lambda(D^2 u)) = g(u) & \text{in } B(x, \tau) \\ \lim_{x \rightarrow x_0} u(x) = +\infty & \forall x_0 \in \partial B(x, \tau) \end{cases}$$

has at least one explosive solution. By construction the solution has radial symmetry. Denote by u_τ the solution obtained by Lemma 2.11. As a consequence of (2.8), we can completely answer the existence question for positive solutions for (1.1). Indeed, applying Lemma 2.8, we have

$$\underline{u} \leq u_n \leq u_\tau \text{ in } B(x, \tau).$$

This entails an upper bound for u_n . In particular, the sequence u_n is uniformly bounded from above in $B(x, \tau/2)$. Notice that any u_n is k -admissible. Pick any compact subset $K \Subset \Omega$. Covering K by finitely many balls $B(x_i, r_i/2)$ we conclude that the sequence $\{u_n\}_n$ is uniformly bounded in K , which is a compact set. Finally, since K is arbitrary chosen it is clear that the limit

$$\lim_{n \rightarrow \infty} u_n(x) = u(x)$$

exists as a continuous function and is a solution of

$$\sigma_k^{1/k}(\lambda(D^2 u)) = g(u) \quad \text{in } \Omega,$$

and therefore $u_n(x) \xrightarrow{n \rightarrow \infty} u(x)$ on any compact subset $K \Subset \Omega$. By construction, $u(x)$ blows up at the boundary. This question is clearly explained in [5] and so u is a boundary blow-up solution of (1.1) in Ω .

It only remains to prove the reverse implication. For $n \in \mathbb{N}^*$, we assume that $u_n > 0$ solve (1.1) in a ball B of centre zero and radius ε_n , where ε_n is a decreasing sequence such that $\varepsilon_n \searrow 0$ as $n \rightarrow \infty$. Proceeding as in [2], we can assume that u_n is a radial symmetric solution. Let $\beta_n = u_n(0)$. We observe that we can assume $\lim_n \beta_n = \infty$, eventually by passing to a subsequence. Then (2.10) follows from (2.20) applied with $\rho = \varepsilon_n$. We finally prove that the sequence $\{\beta_n\}$ is unbounded. If not, up to a subsequence, (β_n) converges to some $\beta \geq 0$. By (2.20) applied with $\rho = \varepsilon_n$, we have

$$0 \leq \int_{\beta_n}^{\infty} \frac{\xi'(r)}{\sqrt[k+1]{(k+1)(G(\xi(r)) - G(\xi(0)))}} dr \leq \frac{(C_{N-1}^{k-1})^{-1/(k+1)}(k+1)}{2k} \varepsilon_n^{\frac{2k}{k+1}}. \quad (3.2)$$

An application of Fubini's theorem together with $n \rightarrow \infty$ in (3.2) leads to

$$\int_{\beta}^{\infty} \frac{\xi'(r)}{\sqrt[k+1]{(k+1)(G(\xi(r)) - G(\xi(0)))}} dr = 0$$

which is not possible.

Remark 3.1. Assume that ψ belongs to a wide class Ψ of monotone increasing convex functions. There is an area in probability theory where boundary-blow-up

problems

$$\begin{cases} \Delta u = \psi(u) & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega \end{cases}$$

arise (see the paper [10] or directly the book [11] for details). The area is known as the theory of superdiffusions, a theory which provides a mathematical model of a random evolution of a cloud of particles. Indeed, given any bounded open set Ω in the N -dimensional Euclidean space, and any finite measure μ we may associate with these the exit measure from Ω i.e. (X_Ω, P_μ) , a random measure which can be constructed by a passage to the limit from a particles system. Particles perform independently Δ -diffusions and they produce, at their death time, a random offspring (cf. [12]). P_μ is a probability measure determined by the initial mass distribution μ of the offspring and X_Ω corresponds to the instantaneous mass distribution of the random evolution cloud. Then proceeding in this way, one can obtain any function ψ from a subclass Ψ_0 of Ψ which contains u^γ with $1 < \gamma \leq 2$. Dynkin [10], also provided a simple probabilistic representation of the solution for the class of problems u^γ ($1 < \gamma \leq 2$), in terms of the so-called exit measure of the associated superprocess. Moreover, the author say that a probabilistic interpretation is known only for $1 < \gamma \leq 2$.

Remark 3.2. The problem of complex Hessian can be easily attacked (see [38] as a starting reference).

REFERENCES

- [1] J. Bao and X. Ji, *Necessary and sufficient conditions on solvability for Hessian inequalities*, Proceedings of the American Mathematical Society, Volume 138, Number 1, January 2010, Pages 175–188.
- [2] J. Bao, X. Ji and H. Li, *Existence and nonexistence theorem for entire subsolutions of k -Yamabe type equations*, J. Differential Equations 253 (2012) 2140–2160
- [3] L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Mathematica, Volume 155, Issue 1, pp 261–301, 1985.
- [4] M. G. Crandall, H. Ishii and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bulletin (New Series) of the American Mathematical Society, Volume 27, Number 1, July 1992.
- [5] D.-P. Covei, *Boundedness and blow-up of solutions for a nonlinear elliptic system*, International Journal of Mathematics, Volume 25, No. 9, Pages 1–12, 2014.
- [6] D.-P. Covei, *Existence of solutions to quasilinear elliptic problems with boundary blow-up*, Annals of the University of Oradea, Fascicola Matematica, Tom XVII, Issue No. 1, Pages 77–84, 2010.
- [7] G. Davila, · P. Felmer and · A. Quaas, *Harnack inequality for singular fully nonlinear operators and some existence results*, Calculus of Variations and Partial Differential Equations, Volume 39, Issue 3–4, pp 557–578, November 2010.
- [8] PH. Delanoe, *Radially symmetric boundary value problems for real and complex elliptic Monge-Ampere equations*, Journal of Differential Equations, Volume 58, Pages 318–344, 1985.
- [9] J. I. Diaz, *Nonlinear Partial Differential Equations and Free Boundaries*, Pitman Research Notes in Mathematics, 106, 1985.
- [10] E.B. Dynkin, *A probabilistic approach to one class of nonlinear differential equations*, Probab. Th. Rel. Fields 89, 89–115 (1991)

- [11] *Selected papers of E. B. Dynkin with commentary*, (E. B. Dynkin; A. A. Yushkevich, G. M. Seitz, A. L. Onishchik, editors).
- [12] E. B. Dynkin, *An Introduction to Branching Measure-Valued Processes*, AMS, Providence, R. I. 1994.
- [13] L. Garding, *An inequality for hyperbolic polynomials*, J. Math. Mech., 8, Pages 957–965, 1959.
- [14] C. Escudero, *On polyharmonic regularizations of k -Hessian equations: Variational methods*, Nonlinear Analysis: Theory, Methods & Applications Volume 125, September 2015, Pages 732–758.
- [15] D. Faraco and X. Zhong, *Quasiconvex functions and Hessian equations*, Archive for Rational Mechanics and Analysis, Volume 168, Pages 245–252, 2003.
- [16] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag Berlin Heidelberg, 1998.
- [17] B. Guan, *The Dirichlet problem for a class of fully nonlinear elliptic equations*, Communications in Partial Differential Equations, Volume 19, Issue 3-4, pages 399-416, January 1994.
- [18] H. Ishii, *On Uniqueness and Existence of Viscosity Solutions of Fully Nonlinear Second-Order Elliptic PDE's*, Communications on Pure and Applied Mathematics, Vol. XLII, 15-45 (1989).
- [19] N M Ivochkina, *The integral method of barrier functions and the Dirichlet problem for equations with operators of Monge-Ampere type*, Mat. Sb. 112 (154) (1980), 193-206; English transl. in Math. USSR Sb. 40 (1981).
- [20] N M Ivochkina, *A description of the stability cones generated by differential operators of Monge-Ampere type*, Mathematics of the USSR-Sbornik, Volume 122, Number 164, 1983 (pp. 265-275; English transl. in Math. USSR Sb. 50, 1985).
- [21] N M Ivochkina, *Description of cones of stability generated by differential operators of Monge - Ampere type*, English transl, Math. USSR Sb. 50, (1985), 259-268.
- [22] N M Ivochkina, *Solution of the Dirichlet problem for some equations of Monge-Ampere type*, Mathematics of the USSR-Sbornik, Volume 56, Number 2, 1987.
- [23] N M Ivochkina, *Solution of the Dirichlet problem for curvature equations of order m* , Mathematics of the USSR-Sbornik, Volume 67, Number 2, 1989.
- [24] N M Ivochkina, *On some properties of the positive m -Hessian operators in $C^2(\Omega)$* , Journal of Fixed Point Theory and Applications, Volume 14, Pages 79–90, 2013.
- [25] N. M. Ivochkina and N. V. Filimonenkova, *On algebraic and geometric conditions in the theory of Hessian equations*, Journal of Fixed Point Theory and Applications, Volume 16, Pages 11–25, 2014.
- [26] H. Jian, *Hessian equations with infinite Dirichlet boundary value*, Indiana University Mathematics Journal, Volume 55, No. 3, Pages 1045–1062, 2006.
- [27] R. Osserman, *On the inequality $\Delta u \geq f(u)$* , Pacific Journal of Mathematics 7 (1957), 1641-1647.
- [28] J. Matero, *Quasilinear elliptic equations with boundary blow-up*, Journal D'Analyse Mathématique, Volume 69, pp. 229-247, 1996.
- [29] J. Matero, *The Bieberbach-Rademacher problem for the Monge-Ampère operator*, Manuscripta Mathematica, Volume 91, 1996, pp. 379-391.
- [30] S. I. Pohozaev (Pokhozhaev), *The Dirichlet problem for the equation $\Delta u = u^2$* , Doklady Acad. Sci. USSR, 136, (1960), no. 3, 769-772. English translation: Soviet. Mathematics Doklady, 1 (1961), 1143-1146.
- [31] J. B. Keller, *On solution of $\Delta u = f(u)$* , Communications on Pure and Applied Mathematics, 10 (1957), 503-510.
- [32] J. B. Keller, *Electrohydrodynamics I. The Equilibrium of a Charged Gas in a Container*, Journal of Rational Mechanics and Analysis, Volume 5, Number 4, 1956.
- [33] P. Salani, *Boundary blow-up problems for Hessian equations*, Manuscripta Mathematica, Volume 96, Pages 281 – 294, 1998.

- [34] N. S. Trudinger and X.-J. Wang, *Hessian measures I*, Topological Methods in Nonlinear Analysis, Volume 10, pp. 225–239, 1997.
- [35] N. S. Trudinger and X.-J. Wang, *Hessian measures II*, Annals of Mathematics, Volume 150, pp. 579–604, 1999.
- [36] J.A. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*, Duke Mathematical Journal, Volume 101, Pages 283–316, 2000.
- [37] J.A. Viaclovsky, *Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds*, Communications in Analysis and Geometry, Volume 10, Pages 815–846, 2002.
- [38] N. Xiang and X.-P. Yang, *The complex Hessian equation with infinite Dirichlet boundary value*, Proceedings of the American Mathematical Society, Volume 136, Number 6, June 2008, Pages 2103–2111.
- [39] B. Wang and J. Bao, *Mirror symmetry for a Hessian over-determined problem and its generalization*, Communications on Pure and Applied Analysis, Volume 13, Number 6, November 2014.

¹DEPARTMENT OF APPLIED MATHEMATICS, THE BUCHAREST UNIVERSITY OF ECONOMIC STUDIES, PIATA ROMANA, 1ST DISTRICT, POSTAL CODE: 010374, POSTAL OFFICE: 22, ROMANIA

E-mail address: coveidragos@gmail.com